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Division of Electromagnetic Research

RESEARCH REPORT No. EM-195

Diffraction of Electromagnetic Waves by a Circular Aperture in an Infinitely Conducting Plane Screen

J. BAZER and L. RUBENFELD

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TABLE OF CONTENTS

	Page
Abstract	ii
Figure	iii
1. Introduction	1
2. Formulation	5
3. Representations of $V_1^{(m)}$ and $V_2^{(m-1)}$ -- E-mode Excitation	14
3.1 The Representations	14
3.2 The Aperture Conditions and the Integral Equations	21
3.3 Ensuring the Proper Edge Behavior	27
4. Explicit Results for Small Values of $\alpha = ka$	30
4.1 General Excitation	30
4.2 Plane Wave Excitation	35
Appendix I	37
Footnotes	40
Reference	42

ABSTRACT

The problem of the diffraction of electromagnetic waves by a circular aperture in an infinitely conducting planar screen is studied. Attention is focused chiefly on the low frequency end of the spectrum where the product ka of the wave number k and the aperture radius a is small. Transverse electric and transverse magnetic modal excitation appropriate to the cylindrical geometry of the problem is assumed. New integral representations of the solution corresponding to incident fields of this kind are presented. These representations, which involve a pair of unknown functions, are designed so as to automatically satisfy Maxwell's equations, the boundary conditions, the radiation condition and all but one of the edge conditions. The unknown functions are shown to satisfy a pair of integro-differential equations which follow directly from the representations by means of an elementary calculation. Convergent power series solutions of these integro-differential equations are readily obtained for small enough ka , when it is assumed merely that the excitation is sufficiently regular in the aperture. These results are employed to derive the first terms in the power series expansions of the far fields, the edge fields, the aperture fields and of the transmission coefficients. When the incident transverse electric and magnetic modes are specialized to those which occur in plane wave excitation these expressions are found to reduce to the well-known plane wave formulas obtained by other means.

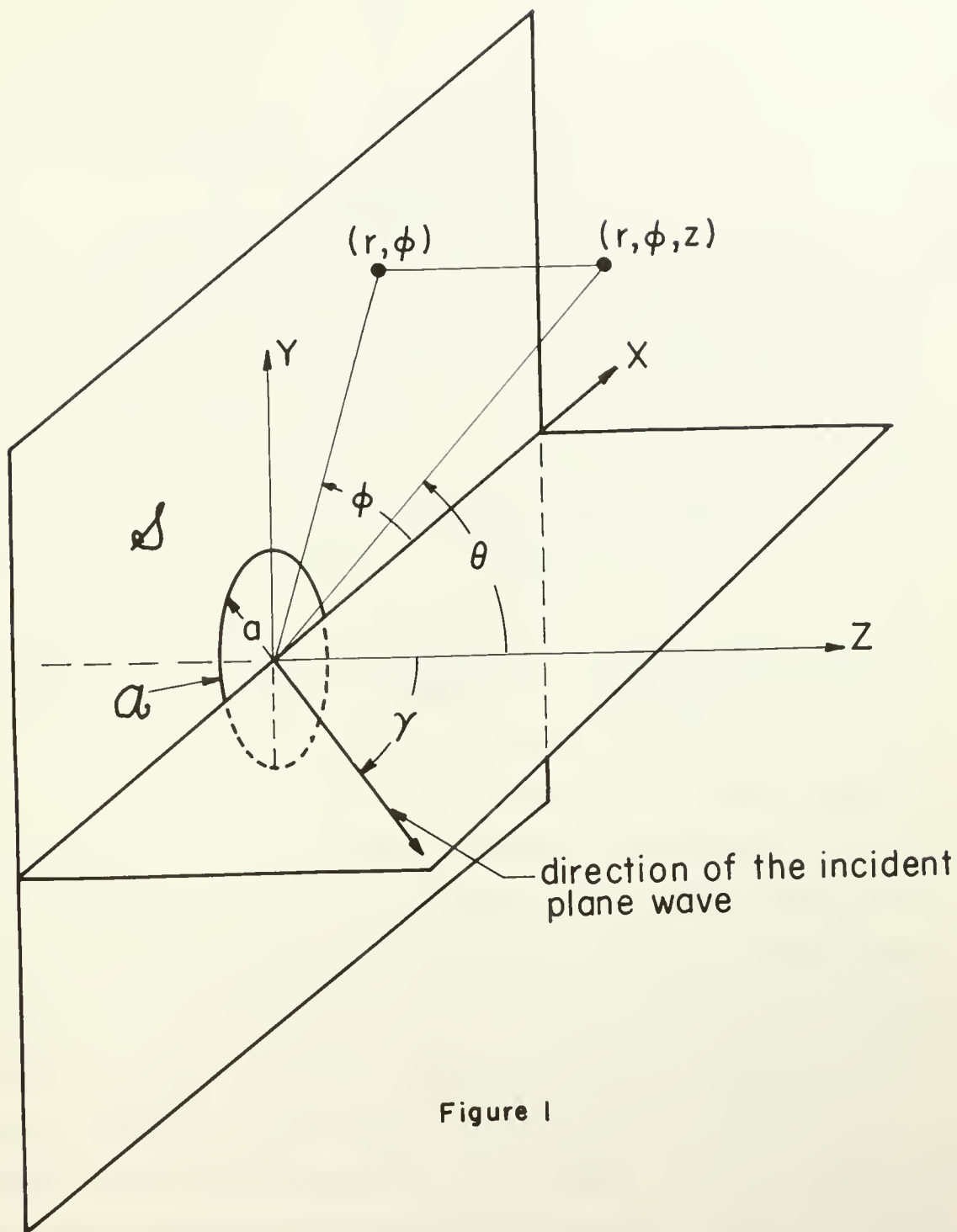


Figure 1

A view of the diffracting screen. $\mathcal{S}$ is the screen, $\mathcal{A}$ is the aperture of radius 'a' centered at the origin of coordinates. $\mathcal{S} + \mathcal{A}$ coincides with the (x,y)-plane. The symbols r , ϕ , and z are cylindrical coordinates of an arbitrary point of space; r denotes the distance from the z -axis (the axis of symmetry); ϕ is measured from the positive x -axis in the (x,y)-plane. The angle θ is measured from the positive z -axis to the ray that connects the origin to (r, ϕ, z) . The angle γ fixes the direction of propagation of the incident plane wave.

1. Introduction

We are concerned with the problem of the diffraction of electromagnetic waves by a circular aperture in a perfectly conducting planar screen when the product ka of the wave number k and the aperture radius a is small. Solutions of this problem have been given by J. Meixner [1]^{*} and J. Meixner and W. Andrejewski [2] using series of spheroidal wave functions, by C. J. Bouwkamp [3a,b] and N. Marcuvitz^{1**} [3b] using formal power series expansions in ka , by H. Levine and J. Schwinger [4] with the aid of variational methods, and by Y. Nomura and S. Katsura [5] and N. Chako [6] employing series expansions of hypergeometric polynomials. A careful survey of the literature on both vector and scalar aperture problems will be found in the report [3b] and paper [3a] by C. J. Bouwkamp.

In this paper, we present an approach to the vector aperture problem which is an extension of the work of J. Bazer and H. Hochstadt [7] on the scalar aperture problem. Our method is most closely related to two recent attacks on the problem due to K. A. Lur'e [8] and W. E. Williams [9b]^{2,3}. All three approaches assume excitation of E-mode or H-mode type such as is derivable by the usual operations from solutions of the reduced wave equation having the form $\cos[m\phi - \phi_O^{(m)}]u(r,z)$, $m = 0, 1, \dots$; here r, z and ϕ denote the cylindrical coordinates of an arbitrary point of space [See Figure 1] and $\phi_O^{(m)}$ is an arbitrary phase angle. In addition, all three approaches have in common the reduction of the vector problem to a pair of scalar aperture problems by means of the Hertz vector potential and all three eventually lead by various means to Fredholm integral equations of the second kind whose

*References are given at the end of the paper.

**Footnotes will be found on pages 41 and 42.

kernels are small when ka is small. The three methods differ among themselves chiefly in their representations of the solutions of the scalar problems mentioned above and in the representations of the vector field to which these give rise. The essentially new element of this paper is to be found in our representations of the scalar and vector fields, which possess the following noteworthy features. First, our representations of the scalar waves have the relatively elementary form⁴

$$(1.1) \quad w_1^{(m)} = \cos[m\phi - \phi_0^{(m)}] r^m \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^m \int_{-1}^1 \frac{e^{ik R^+(r,z;t)}}{R^+(r,z;t)} f_1^{(m)}(t) dt,$$

$$(1.2) \quad w_2^{(m-1)} = \cos[m\phi - \tilde{\phi}_0^{(m)}] r^{|m-1|} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{|m-1|} \int_{-1}^1 \frac{e^{ik R^+(r,z;t)}}{R^+(r,z;t)} f_2^{(m-1)}(t) dt,$$

$$m = 0, 1, \dots$$

where $R^+ = \sqrt{r^2 + (z + iat)^2}$, $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ are odd and even functions, respectively, of a certain general class and $\tilde{\phi}_0^{(m)} = \phi_0^{(m)}$ or $\tilde{\phi}_0^{(m)} = \phi_0^{(m)} + \frac{\pi}{2}$. Second, when ka is small, they provide the basis for the efficient calculation of explicit formulas for the field everywhere in space and for the various field characteristics such as the aperture fields, far fields and transmission coefficients for all $m \geq 0$. Third, only the most elementary analysis is required in dealing with these representations; no use is made, for example, of special function theory. Finally, the field representations derived from $w_1^{(m)}$ and $w_2^{(m-1)}$ have certain built in features which significantly simplify the task of meeting the conditions of the problem. Specifically, the field representations are designed to satisfy automatically Maxwell's equations,

the boundary and the radiation condition assuming merely that the unknown functions $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ exist. Then, the condition that the magnetic field be continuous through the aperture leads directly, by means of an elementary calculation, to a pair of linear integro-differential equations⁵ for $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ of a type already discussed by Bazer and Hochstadt [7] in their treatment of the scalar aperture problem. Because the inhomogeneous terms of these integro-differential equations depend on an arbitrary constant (which appears linearly in both terms) the reduction of the vector problem to a pair of scalar problems is not, however, a complete one -- the scalar problems are coupled. Nevertheless, one obtains with relatively little labor, convergent power series expansions in ka for $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ when ka is sufficiently small just as in the scalar case. The solution of the problem is then completed, as in references [8], [9] and [10], by adjusting the arbitrary constant so as to ensure the proper edge behavior. It may be mentioned that the derivations in [8], [9a, 9b], [10] of power series expansions of the analogs of $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ are relatively less direct than the one which we have just described.

By the method just outlined, we have calculated for each $m \geq 0$ the first few terms of the power series for $f_1^{(m)}$ and $f_2^{(m-1)}$ corresponding⁶ to rather general classes of E- and H-mode excitation. We have nevertheless been able to derive for each $m \geq 0$ the first terms⁶ of the power series for the electric field in the aperture, the edge fields, the far field amplitudes and the transmission coefficients. When the E- and H-mode fields are specialized to those which appear in the modal expansions of plane waves

our formulas for $m = 0,1$ reduce to results obtained by other means [see references [3a, b]]. Our formulas for large m seem however to be new although it should be possible to derive them from Williams' results [9b].

2. Formulation

The geometrical setup is shown in Figure 1. $\mathcal{A}$ and $\mathcal{S}$ denote the aperture and screen respectively; a is the radius of the aperture. Together $\mathcal{A}$ and $\mathcal{S}$ make up the x,y -plane. The origin is located at the center of $\mathcal{A}$ and the z -axis is perpendicular to the plane of the screen. For the most part we employ the cylindrical coordinates (r,ϕ,z) ,

$$(2.1) \quad \begin{aligned} x &= r \cos \phi, \\ y &= r \sin \phi, \quad 0 \leq \phi \leq 2\pi, \\ z &= z, \end{aligned}$$

r denoting the distance from the z -axis. In discussing the far fields however, it is more convenient to employ the spherical coordinates (R,θ,ϕ) , where

$$(2.2) \quad \begin{aligned} x &= R \sin \theta \cos \phi, \\ y &= R \sin \theta \sin \phi, \quad 0 \leq \phi \leq 2\pi, \\ z &= R \cos \theta, \quad 0 \leq \theta \leq \pi, \\ R &= \sqrt{r^2 + z^2} \end{aligned}$$

The angle θ is the angle between the positive z -axis and the position vector to the point (x,y,z) .

In the case of cylindrical coordinates $\underline{e}_r, \underline{e}_\phi, \underline{e}_z$ denote unit vectors at (r,ϕ,z) in the r, ϕ and z directions; in the case of spherical coordinates $\underline{e}_R, \underline{e}_\theta, \underline{e}_\phi$ denote unit vectors at (R,θ,ϕ) along the radial, latitudinal and meridional directions.

Assuming that the excitation originates in the region $z < 0$, we represent the total field $(\underline{E}, \underline{H})$ as follows:

$$(2.3) \quad \begin{aligned} z < 0: & \quad \begin{cases} \mathcal{E}(\underline{z}) = \underline{E}_0(\underline{z}) - \underline{E}_0^*(\underline{z}^*) + \underline{E}^*(\underline{z}), \\ \mathcal{H}(\underline{z}) = \underline{H}_0(\underline{z}) + \underline{H}_0^*(\underline{z}^*) - \underline{H}^*(\underline{z}), \end{cases} \\ z > 0: & \quad \begin{cases} \mathcal{E}(\underline{z}) = \underline{E}(\underline{z}), \\ \mathcal{H}(\underline{z}) = \underline{H}(\underline{z}). \end{cases} \end{aligned}$$

Here, $(\underline{E}_0(\underline{z}), \underline{H}_0(\underline{z}))$ is the incident field, $(\underline{E}_0(\underline{z}) - \underline{E}_0^*(\underline{z}^*), \underline{H}_0(\underline{z}) + \underline{H}_0^*(\underline{z}^*))$ is the incident and reflected field that would be present in the absence of an aperture, $(\underline{E}(\underline{z}), \underline{H}(\underline{z}))$ is the (unknown) scattered field in the region $z > 0$ and $(\underline{E}^*(\underline{z}^*), -\underline{H}^*(\underline{z}^*))$ is the back-scattered field in the region $z < 0$. An asterisk signifies reflection in the z -plane; thus if $\underline{Q} = (Q_x, Q_y, Q_z)$ denotes any Cartesian vector then by definition $\underline{Q}^* = (Q_x, Q_y, -Q_z)$.

We assume E and H-mode excitation of the form

E-mode (Transverse Magnetic)

$$(2.4) \quad \begin{aligned} \text{a) } \underline{E}_{0,t} &= \nabla_t \left(\sin m\phi_m \frac{\partial V_0^{(m)}}{\partial z} \right), & m\phi_m &\equiv m\phi - \phi_0^{(m)}, \quad (m = 0, 1, 2, \dots), \\ \text{b) } \underline{H}_{0,t} &= ik \underline{e}_z \wedge \nabla_t (\sin m\phi_m V_0^{(m)}), \end{aligned}$$

$$\text{c) } E_{0,z} = -\nabla_t^2 (\sin m\phi_m V_0^{(m)}) , \quad H_{0,z} \equiv 0,$$

where $V_0^{(m)} = V_0^{(m)}(r, z)$ satisfies

$$\text{d) } \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r \partial V_0^{(m)}}{\partial r} \right) + \frac{\partial^2 V_0^{(m)}}{\partial z^2} + \left[k^2 - \frac{m^2}{r^2} \right] V_0^{(m)} = 0;$$

H-mode (Transverse Electric)

$$\begin{aligned}
 (2.5) \quad & \text{a) } \underline{E}_{0,t} = ik \underline{e}_z \wedge \nabla_t (\cos m\bar{\phi}_m \bar{V}_0^{(m)}) , \quad m\bar{\phi}_m \equiv m\phi - \bar{\phi}_0^{(m)}, \quad (m = 0, 1, 2, \dots) \\
 & \text{b) } \underline{H}_{0,t} = -\nabla_t \left(\cos m\bar{\phi}_m \frac{\partial \bar{V}_0^{(m)}}{\partial z} \right) , \\
 & \text{c) } E_{0,z} \equiv 0, \quad H_{0,z} = \nabla_t^2 (\cos m\bar{\phi}_m \bar{V}_0^{(m)}) ,
 \end{aligned}$$

where $\bar{V}_0^{(m)} = \bar{V}_0^{(m)}(r, z)$ satisfies

$$\text{d) } \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{V}_0^{(m)}}{\partial r} \right) + \frac{\partial^2 \bar{V}_0^{(m)}}{\partial z^2} + \left[k^2 - \frac{m^2}{r^2} \right] \bar{V}_0^{(m)} = 0.$$

In these equations $\phi_0^{(m)}$ and $\bar{\phi}_0^{(m)}$ are arbitrary phase angles, ∇_t is the operator $\nabla - \underline{e}_z \frac{\partial}{\partial z}$ (∇ is the gradient operator in three dimensions) and $\underline{Q}_t = \underline{Q} - \underline{e}_z Q_z$, $\underline{Q} = (Q_x, Q_y, Q_z)$ being any vector.

We represent the scattered field in terms of the Hertz vector potential $\underline{\Pi}$,

$$\begin{aligned}
 (2.6) \quad & E = ik \nabla \wedge \underline{\Pi} \\
 & \underline{H} = k^2 \underline{\Pi} + \nabla(\nabla \cdot \underline{\Pi}),
 \end{aligned}$$

where (in Cartesian coordinates) $\underline{\Pi}$ satisfies

$$(2.7) \quad \nabla^2 \underline{\Pi} + k^2 \underline{\Pi} = 0.$$

Assuming E-mode incidence and setting [cf. references [8], [9], [10]]

$$\begin{aligned}
 \Pi_x &= \cos[(m-1)\phi - \phi_0^{(m)}] V_2^{(m-1)}(r, z), \\
 \Pi_y &= -\sin[(m-1)\phi - \phi_0^{(m)}] V_2^{(m-1)}(r, z), \\
 \Pi_z &= \cos[m\phi - \phi_0^{(m)}] V_1^{(m)}(r, z), \quad m = 0, 1, 2, \dots,
 \end{aligned}
 \tag{2.8}$$

where $V_1^{(m)}(r, z)$ and $V_2^{(m-1)}(r, z)$ denote two unknown scalar functions of r and z we find immediately that the radial and ϕ -components of $\underline{\Pi}$ are

$$\begin{aligned}
 \Pi_r &= \cos m\phi_m V_2^{(m-1)}(r, z), & m\phi_m &= m\phi - \phi_0^{(m)} \\
 \Pi_\phi &= -\sin m\phi_m V_2^{(m-1)}(r, z), & m &= 0, 1, 2, \dots
 \end{aligned}
 \tag{2.9}$$

from which it follows, owing to equation (2.6), that the corresponding components of $\underline{E}$ are

$$\begin{aligned}
 E_r &= ik \left\{ \frac{\partial V_2^{(m-1)}}{\partial z} - \frac{m}{r} V_1^{(m)} \right\} \sin m\phi_m, \\
 E_\phi &= ik \left\{ \frac{\partial V_2^{(m-1)}}{\partial z} - \frac{\partial}{\partial r} V_1^{(m)} \right\} \cos m\phi_m.
 \end{aligned}
 \tag{2.10}$$

To obtain the corresponding H-mode formulas we have only to replace $V_2^{(m-1)}$, $V_1^{(m)}$, $\phi_0^{(m)}$ and ϕ_m by $\bar{V}_2^{(m-1)}$, $\bar{V}_1^{(m)}$, $\bar{\phi}_0^{(m)}$ and $\bar{\phi}_m$ in equations (2.8)-(2.10) (see footnote 7).

Suppose now that we are successful in finding functions Π_x , Π_y and Π_z of the form of the right members of equation (2.8) having the following properties:

- (i) Π_x , Π_y , Π_z satisfy the reduced scalar wave equation.

(ii) Screen conditions

$$(2.11) \quad 0 = \frac{\partial V_2^{(m-1)}}{\partial z} = V_1^{(m)} \quad \text{on } \mathcal{S}$$

(iii) Radiation conditions: as R approaches infinity

$$(2.12) \quad \begin{aligned} V_1^{(m)} &\sim A_1^{(m)}(\theta) \frac{e^{ikR}}{R}, \\ V_2^{(m-1)} &\sim A_2^{(m-1)}(\theta) \frac{e^{ikR}}{R}, \end{aligned} \quad m = 0, 1, 2, \dots$$

(iv) Edge conditions: $V_1^{(m)}$ and $V_2^{(m-1)}$ are such that

$$(2.13) \quad \frac{\partial V_2^{(m-1)}}{\partial z} - \frac{\partial V_1^{(m)}}{\partial r} = \mathcal{O}(\rho^{1/2}),$$

$$\frac{\partial V_2^{(m-1)}}{\partial z} - \frac{m}{r} V_2^{(m)} = \mathcal{O}(\rho^{-1/2}),$$

in the aperture $\mathcal{A}$ at a distance ρ from the circular edge.

(v) The aperture condition.⁷ If the excitation is of E-mode type then in the aperture $\mathcal{A}$ ($r < a$)

$$(2.14) \quad \frac{\partial V_1^{(m)}}{\partial z} = c_m k (kr)^m - \frac{i}{k} \frac{\partial^2 V_o^{(m)}}{\partial z^2},$$

$$V_2^{(m-1)} = -mc_m (kr)^{m-1} - ik^{-1} r^{-m} \frac{\partial}{\partial r} (r^m V_o^{(m)})$$

If the excitation is of H-mode type then in the aperture $\mathcal{A}$ ($r < a$)

$$(2.15) \quad \begin{aligned} \bar{V}_1^{(m)} &= - \frac{\partial \bar{V}_0^{(m)}}{\partial z} - \bar{c}_m k(kr)^m, \\ \bar{V}_2^{(m-1)} &= m \bar{c}_m (kr)^{m-1}. \end{aligned}$$

In equation (2.14) and (2.15) c_m and $\bar{c}_m$ are arbitrary constants which will later be employed to satisfy the edge condition (iv).

We maintain that if such functions can be found, then we have in effect solved the vector problem. For if condition (i) is satisfied then $\underline{\Pi}$ satisfies the equation (2.7) and the derived quantities $(\underline{E}, \underline{H})$ of equation (2.6) satisfy Maxwell's equations. As a mere inspection of equation (2.10) reveals, (ii) and (iv) are simply conditions which are sufficient to ensure that $\underline{E}_t$ vanishes on $\mathcal{S}$ and has the proper edge behavior⁸. Furthermore, the radiation condition (iii) and equation (2.6) imply after a brief calculation that as $R \rightarrow \infty$

$$(2.16) \quad \begin{aligned} \underline{E} &= -k^2 \left\{ \underline{e}_R \wedge \underline{\tilde{A}}^{(m)}(\theta, \phi_m) \right\} \frac{e^{ikR}}{R} + \mathcal{O}(R^{-2}), \\ \underline{H} &= -k^2 \left\{ \underline{e}_R \wedge (\underline{e}_R \wedge \underline{\tilde{A}}^{(m)}(\theta, \phi_m)) \right\} \frac{e^{ikR}}{R} + \mathcal{O}(R^{-2}), \end{aligned}$$

where

$$(2.17) \quad \begin{aligned} \underline{\tilde{A}}^{(m)}(\theta; \phi_m) &= (A_2^{(m-1)} \sin \theta + A_1^{(m)} \cos \theta) \cos m\phi_m \underline{e}_R \\ &\quad + (A_2^{(m-1)} \cos \theta - A_1^{(m)} \sin \theta) \cos m\phi_m \underline{e}_\theta \\ &\quad - A_2^{(m-1)} \sin m\phi_m \underline{e}_\phi. \end{aligned}$$

It follows that $R|\underline{E}|$ and $R|\underline{H}|$ are bounded when $R \rightarrow \infty$ and that

$$(2.18) \quad 0 = \lim_{R \rightarrow \infty} \left\{ R[\underline{e}_R \wedge \underline{\tilde{E}} - \underline{\tilde{H}}] \right\} = \lim_{R \rightarrow \infty} \left\{ R[\underline{e}_R \wedge \underline{\tilde{H}} - \underline{\tilde{E}}] \right\}$$

which are the appropriate radiation conditions for the electromagnetic field [see reference [14]]. Finally, as we show in Appendix I, the aperture conditions are equivalent to the requirement that $\mathcal{H}_t$ be continuous through the aperture. It is a well-known fact that this condition guarantees the continuity of $\underline{\mathcal{E}}$ and the first derivatives of $(\underline{\mathcal{E}}, \underline{\mathcal{H}})$ through the aperture provided that $\mathcal{H}_t$ is sufficiently differentiable in $\mathcal{A}$.

For technical reasons of this kind it is necessary to place restrictions on the incident field in the aperture. The following conditions are sufficient to ensure the success of the ensuing analysis but they are by no means necessary conditions. Assuming E-mode incidence, we require that $V_o^{(m)}$, $\partial V_o^{(m)} / \partial z$ and $\partial^2 V_o^{(m)} / \partial z^2$ have the following form in the aperture $\mathcal{A}$.

$$(2.19) \quad \begin{aligned} V_o^{(m)} &= (kr)^m \sum_{\mu=0}^{\infty} h_{\mu}^{(0)(m)} (kr)^{2\mu}, \\ \frac{\partial V_o^{(m)}}{\partial z} &= k(kr)^m \sum_{\mu=0}^{\infty} h_{\mu}^{(1)(m)} (kr)^{2\mu}, \quad (\text{E-mode}) \\ \frac{\partial^2 V_o^{(m)}}{\partial z^2} &= k^2 (kr)^m \sum_{\mu=0}^{\infty} h_{\mu}^{(2)(m)} (kr)^{2\mu}, \end{aligned}$$

and that⁹

$$(2.20) \quad \frac{\partial V_o^{(m)}}{\partial z} = k\beta^{(m)} V_o^{(m)} \quad (\text{in } \mathcal{A})$$

where $\beta^{(m)}$ is a constant. The three series are assumed to converge in a circular domain of radius $a + \Delta$, $\Delta > 0$ centered at $r = 0$; the coefficients $h_{\mu}^{(j)}$, $j = 0, 1, 2$; $\mu = 1, 2, \dots$ are assumed to be independent of k but may depend on m . The corresponding requirements for H-mode incidence may be obtained from the above simply by replacing $V_O^{(m)}$, $h_{\mu}^{(j)}$ and $\beta^{(m)}$ by $\bar{V}_O^{(m)}$, $\bar{h}_{\mu}^{(j)}$ and $\bar{\beta}^{(m)}$, respectively.

Suppose the incident field is a plane wave whose direction of propagation $\underline{n}$ is in the (x, z) -plane and makes an angle $\gamma < \pi/2$ with the positive z -axis. Such an incident field may be expressed as a superposition of the plane waves

$$(2.21) \quad \begin{aligned} (a) \quad \underline{E}_O &= -\underline{e}_y \wedge \underline{n} k^2 \exp(ik(\underline{n} \cdot \underline{x})), & \underline{H}_O &= -\underline{e}_y k^2 \exp(ik\underline{n} \cdot \underline{x}); \\ (b) \quad \bar{\underline{E}}_O &= -\underline{e}_y k^2 \exp(ik\underline{n} \cdot \underline{x}), & \bar{\underline{H}}_O &= +\underline{e}_y \wedge \underline{n} k^2 \exp(ik\underline{n} \cdot \underline{x}) \end{aligned}$$

where $\underline{e}_y$ is the unit vector parallel to the y -axis. Using the Fourier-Bessel series expansion of the first factor of $\exp(ikr \sin \gamma \cos \phi) \exp(ikz \cos \gamma) = \exp(ik\underline{n} \cdot \underline{x})$ we find that

$$(2.22) \quad V_O^{(m)} = \bar{V}_O^{(m)} = \frac{i^m \kappa_m J_m(kr \sin \gamma) e^{ikz \cos \gamma}}{\sin \gamma}$$

$$(\kappa_m = 1 \text{ when } m = 0; \quad \kappa_m = 2 \text{ when } m = 1, 2, \dots)$$

and that for all m , including $m = 0$, the products $(m\phi_m)$ and $(m\bar{\phi}_m)$ have the values

$$(2.23) \quad (m\phi_m) = m\phi + \frac{\pi}{2}; \quad (m\bar{\phi}_m) = m\phi.$$

The functions $V_o^{(m)}$ and $\bar{V}_o^{(m)}$ evidently satisfy all the requirements listed above and from the power series representation of $J_m(kr \sin \gamma)$ we find easily that

$$h_{\mu}^{(0)}(m) = \bar{h}_{\mu}^{(0)}(m) = \frac{i^{m+2\mu} \kappa_m (\sin \gamma)^{m+2\mu-1}}{2^{m+2\mu} \mu! \Gamma(m+\mu+1)},$$

$$(2.25) \quad h_{\mu}^{(1)}(m) = \bar{h}_{\mu}^{(1)}(m) = \frac{i^{m+2\mu+1} \kappa_m \cos \gamma (\sin \gamma)^{m+2\mu-1}}{2^{m+2\mu} \mu! \Gamma(m+\mu+1)}$$

$$h_{\mu}^{(2)}(m) = \bar{h}_{\mu}^{(2)}(m) = \frac{i^{m+2\mu+2} \kappa_m \cos^2 \gamma (\sin \gamma)^{m+2\mu-1}}{2^{m+2\mu} \mu! \Gamma(m+\mu+1)}$$

and that

$$(2.25) \quad \beta^{(m)} = \bar{\beta}^{(m)} = i \cos \gamma$$

3. Representations of $V_1^{(m)}$ and $V_2^{(m-1)}$ - E-Mode Excitation.¹⁰

3.1 The Representations. In this subsection, we describe the representations of $V_1^{(m)}$ and $V_2^{(m-1)}$ associated with E-mode incidence and prove in detail that conditions (i)-(v) of Section 2 can be satisfied. The corresponding analysis for the case of H-mode incidence is completely analogous and will therefore be omitted.

Following J. Bazer and A. Brown [11a,b], W. D. Collins [12] and J. Bazer and H. Hochstadt [7], we represent $V_1^{(m)}$ and $V_2^{(m-1)}$ by the expressions¹¹

$$\begin{aligned}
 (3.1) \quad V_j^{(m)} &= r^{|m|} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{|m|} \int_{-1}^1 \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} f_j^{(m)}(t) dt \\
 &= r^{|m|} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{|m|} \int_0^1 \left\{ \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} + (-1)^j \frac{e^{ikR^-(r,z;t)}}{R^-(r,z;t)} \right\} f_j^{(m)}(t) dt
 \end{aligned}$$

for $j = 1, 2$. where

$$(3.2) \quad R^\pm(r, z; t) = \sqrt{r^2 + (z \pm iat)^2}.$$

Hereafter, it is to be understood that m may be any non-negative integer when $j = 1$ and that $m = -1$ or any non-negative integer when $j = 2$. To fulfill the requirements on $V_1^{(m)}$ and $V_2^{(m)}$ we assume that the $f_j^{(m)}(t)$ have the following properties:

(i) $f_j^{(m)}(t)$ is even or odd according as j is even or odd; we have already made use of this property in the second expression for the $V_j^{(m)}$.

(ii) $f_1^{(m)}(t)$ and $f_2^{(m)}(t)$ are regular in t in the circular domain defined by the relation $|t| \leq 1+\Delta/a$, $\Delta > 0$ for all sufficiently small values of ka and all $^{12} m \geq 1$. Since the regularity of the incident field in the domain $a + \Delta$, $\Delta > 0$ [see the remarks following equation (2.20)] ultimately imply that $f_j^{(m)}(t)$, $j = 1, 2$ have this property, no generality is lost in assuming it at the outset. Finally, to ensure the proper edge behavior [see equation (2.13)] it turns out to be sufficient to require

(iii) that for each $m \geq 1$ ($D = \frac{d}{dt}$)

$$(3.3) \quad \begin{aligned} D^k f_1^{(m)}(1) &= 0, & k &= 0, 1, \dots, m-1 \\ D^k f_2^{(m-1)}(1) &= 0, & k &= 0, 1, \dots, m-2 \end{aligned}$$

and that

$$(3.4) \quad D^{|m-1|} f_2^{(m-1)}(1) - \frac{1}{ia} D^m f_1^{(m)}(1) = 0.$$

When $m = 0$ condition (3.3) is to be ignored.

To resolve the ambiguity in the definition of $R^\pm(r, z; t)$ when $z = 0$, we assume that

$$(3.5) \quad \lim_{z \downarrow 0} R^\pm(r, z; t) = \sqrt{r^2 - a^2 t^2}, \quad \text{when } r^2 > a^2 t^2$$

which implies, on choosing the branch of $R^\pm = \sqrt{r^2 - (at \mp iz)^2}$ in an appropriate manner, that

$$(3.5') \quad \lim_{z \downarrow 0} R^\pm(r, z; t) = \pm i \sqrt{a^2 t^2 - r^2}, \quad \text{when } r^2 < a^2 t^2.$$

At points in the aperture, ($z = 0$, $r < a$) the only points in physical space where the integrands of equation (3.1) may be singular, $V_j^{(m)}$, $j = 1, 2$ and its first and higher order derivatives are defined to be the limits

approached when z approaches zero through positive values. The limits themselves can be rigorously derived with the aid of the properties of the $f_j^{(m)}(t)$ listed above and standard function-theoretic techniques. Appendix I of reference [11a] relating to the calculation of the edge behavior exemplifies in detail the sort of argument required.¹³ Here, it is enough to say that the several limiting processes employed in what follows are justifiable and that their end results agree with those that are obtained by purely formal considerations.

If it is assumed merely that the $f_j^{(m)}(t)$, $j = 1, 2$, satisfying (i)-(iii) above exist, then it can be shown even at this stage that $V_1^{(m)}$ and $V_2^{(m-1)}$ satisfy all the requirements listed in Section 2 with the exception of the aperture condition. We must show first that $\Pi_z = \cos m\phi V_1^{(m)}(r, z)$ and $(\Pi_x, \Pi_y) = (\cos(m-1)\phi_m, -\sin(m-1)\phi_m) V_2^{(m-1)}(r, z)$ with $V_1^{(m)}$ and $V_2^{(m-1)}$ defined as in equation (3.1) satisfy the scalar wave equation

$$(3.7) \quad \frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial y^2} + \frac{\partial^2 \Pi}{\partial z^2} + k^2 \Pi = 0.$$

The proof is essentially the same for Π_x , Π_y and Π_z so that it will be sufficient to explain it in relation to one of these components -- Π_z say. Now let $W(x, y, z)$ be any solution of equation (3.7). Then $\partial W / \partial x$ and $\partial W / \partial y$ are also solutions and hence so are the functions.

$$(3.8) \quad D^\pm W = \left(\frac{\partial}{\partial y} \pm i \frac{\partial}{\partial x} \right) W.$$

In fact,

$$(3.9) \quad (D^\pm)^m W = \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right)^m W$$

are solutions of equation (3.7), $m = 1, 2, \dots$. Expressing $D^\pm$ in polar co-ordinates [see equation (2.1)] we arrive at the operator identity

$$(3.10) \quad D^\pm = e^{\pm i\phi} \left(\frac{\partial}{\partial r} \pm i \frac{\partial}{\partial \phi} \right).$$

If $D^\pm$ is now applied to any axially symmetric solution $W = U(r, z)$ of the wave equation it follows easily that

$$(D^\pm)^m U = e^{\pm im\phi} \left[\frac{\partial}{\partial r} - \frac{m-1}{r} \right] \left[\frac{\partial}{\partial r} - \frac{m-2}{r} \right] \dots \left[\frac{\partial}{\partial r} - \frac{1}{r} \right] \frac{\partial}{\partial r} U(r, z)$$

which may be written more compactly as

$$(3.11) \quad (D^\pm)^m U(r, z) = e^{\pm im\phi} r^m \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^m U(r, z).$$

Evidently, any linear combination of $(D^+)^m U$ and $(D^-)^m U$ is a solution of the scalar wave equation provided U is. If we set

$$(3.12) \quad U(r, z) = \int_{-1}^1 \frac{e^{ikR^+(r, z; t)}}{R^+(r, z; t)} f_j^{(m)}(t) dt$$

then, since U is obviously a solution of the scalar wave equation, so is the following linear combination of $(D^+ U)^m$ and $(D^- U)^m$

$$(3.13) \quad \Pi_z = \cos[m\phi - \phi_0^{(m)}] r^m \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^m \int_{-1}^1 \frac{e^{ikR^+(r, z; t)}}{R^+(r, z; t)} f_j^{(m)}(t) dt.$$

Condition (i) of Section 2 is thus satisfied.

To verify that $\partial V_2^{(m-1)} / \partial z$ and $V_1^{(m)}$ vanish on $\mathcal{S}$ [condition (ii)] we need only note that the integrands in the last member of equations (3.1) and their first derivatives are non-singular when $r > a$; the vanishing of

$\partial V_2^{(m-1)}/\partial z$ and $V_1^{(m)}$ then follows immediately on letting z approach zero under the integral signs.

To verify the radiation condition (condition iii) we make use of the spherical polar coordinates of equation (2.2) and observe that

$$R^\pm(r, z; t) = R \pm iat \cos \theta + O(R^{-1})$$

when R is large. With the aid of this formula it follows immediately that equations (2.12) are satisfied, $A_1^{(m)}(\theta)$ and $A_2^{(m-1)}(\theta)$ being defined by

$$(3.14) \quad A_1^{(m)}(\theta) = -2(ik \sin \theta)^m \int_0^1 \sinh(\alpha t \cos \theta) f_1^{(m)}(t) dt,$$

$$(3.15) \quad A_2^{(m-1)}(\theta) = 2(ik \sin \theta)^{|m-1|} \int_0^1 \cosh(\alpha t \cos \theta) f_2^{(m-1)}(t) dt$$

To prove that the edge conditions (iv) of Section 2 are satisfied, it is necessary to determine the behavior of the functions $\partial V_2^{(m-1)}/\partial z$, $V_1^{(m)}$ and $\partial V_1^{(m)}/\partial r$ in the neighborhood of the circular edge. To exemplify our procedure, we show how the behavior of $V_1^{(m)}$ is obtained. We begin by assuming that $|z/a| < \epsilon$, where $\epsilon < r/a$ is a small positive number. Then we deform the contour near $t = 0$ in the right member of equation (3.1) into a semi-circle of radius ϵ centered at $t = 0$ and falling below the real t -axis. On the new contour -- C_ϵ say -- the ratio $|z/a|$ will then be everywhere less than $|t|$. Then, we observe that the integral representation of $V_1^{(m)}$ [see equation (3.1)] may be expressed as

$$(3.16) \quad v_1^{(m)} = \left(\frac{ir}{a}\right)^m \int_{C_\epsilon} \frac{\exp\{ik R^+(r, z; t)\}}{R^+(r, z; t)} \left[\frac{d}{dt} \left(\frac{1}{z+iat}\right)\right]^m f_1^{(m)}(t) dt,$$

on noting that

$$(3.17) \quad \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\exp(ikR^+)}{R^+} \right) = \frac{1}{ia(z+iat)} \frac{d}{dt} \left(\frac{\exp(ikR^+)}{R^+} \right),$$

integrating by parts m times and finally making use of the behavior of $f_1^{(m)}(t)$ at $t = \pm 1$ [see equation (3.3), and recall that $f_1^{(m)}(t)$ is odd].

Next, we make use of the fact that $\exp(ikR^+)/R^+ = (z+iat)^{-1} d(\exp ikR^+)/dt$, integrate by parts once again and find that

$$(3.18) \quad v_1^{(m)} = -\frac{1}{ka} \left(\frac{ir}{a}\right)^m \left\{ \frac{\exp\{ikR^+(r, z; t)\}}{z+iat} \left[\frac{d}{dt} \left(\frac{1}{z+iat}\right)\right]^m f_1^{(m)}(t) \right\} \Big|_{-1}^1 \\ - \int_{C_\epsilon} \frac{\exp\{ikR^+(r, z; t)\}}{R^+(r, z; t)} \left[\frac{d}{dt} \left(\frac{1}{z+iat}\right)\right]^{m+1} f_1^{(m)}(t) dt.$$

If we set $z = 0$ and write

$$(3.19) \quad r = a - \rho, \quad \rho \ll a$$

and expand R^+ and $\exp(ik R^+)$ in powers of ρ/a we find that¹⁴ in the aperture $\mathcal{A}$

$$(3.20) \quad v_1^{(m)} = \frac{2}{ia^{m+1}} \frac{d^m f_1^{(m)}(1)}{dt^m} \left[\sqrt{2} (\rho/a)^{1/2} + \mathcal{O}[(\rho/a)^{3/2}] \right].$$

Similarly, we find that¹⁵ in $\mathcal{A}$

$$(3.21) \quad \frac{\partial V_1^{(m)}}{\partial \rho} = \frac{1}{i a^{m+2}} \frac{d^{(m)} f_1^{(m)}(1)}{dt^m} \left[\sqrt{2}(\rho/a)^{-1/2} + \mathcal{O}[(\rho/a)^{1/2}] \right],$$

$$(3.22) \quad \frac{dV_2^{(m-1)}}{dz} = \frac{1}{a} \left(\frac{1}{a^m} \right) \frac{d^{m-1} f_2^{m-1}(1)}{dt^{m-1}} \left[-\sqrt{2}(\rho/a)^{-1/2} + \mathcal{O}[(\rho/a)^{1/2}] \right].$$

It can also be shown, in the same way, that [see reference [7]]

$$(3.23) \quad V_1^{(m)} = \frac{2\sqrt{2}}{i a^{m+1}} \frac{d^m f_1^{(m)}(1)}{dt^m} (\rho/a)^{1/2} \sin(\delta/2) + \mathcal{O}[(\rho/a)]$$

$$(3.24) \quad V_2^{(m-1)} = V_2^{(m-1)} \Big|_{\substack{r=1 \\ z \downarrow 0}} - 2\sqrt{2} \left(\frac{1}{a^m} \right) \frac{d^{|m-1|}}{dt^{|m-1|}} f_2^{m-1}(1) (\rho/a)^{1/2} \cos \frac{\delta}{2} + \mathcal{O}[(\rho/a)]$$

Here, we have put

$$(3.25) \quad \begin{aligned} z &= \rho \sin \delta, \\ r &= a + \rho \cos \delta, \end{aligned} \quad 0 \leq \delta \leq \pi.$$

[see Figure 2]. $V_2^{(m-1)} \Big|_{\substack{r=1 \\ z \downarrow 0}}$ is a constant whose value may be calculated from equation (2.14) once the constant c_m has been fixed. $V_1^{(m)}$ and $V_2^{(m-1)}$ in equations (3.24) and (3.25) may be differentiated formally with respect to ρ and δ ; the resulting formulas then lead to the correct expressions for $(\underline{E}, \underline{H})$ in the neighborhood of the edge.

Before outlining how $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ are determined, we should mention that there are several ways to represent the solution of the scalar

problems formulated in Section 2. In addition to the works already cited at the beginning of this section mention should be made of the works of D. S. Jones [13a], A. E. Heins and R. C. MacCamy [15] and W. E. Williams [9b]. How these methods are related to each other is discussed by B. Noble in [16] and by J. Bazer and H. Hochstadt in the appendix of [7].

3.2 The Aperture Conditions and the Integral Equations.

We begin by computing the limiting value of $\partial V_1^{(m)} / \partial z$ in the aperture. Assuming $z > 0$, we differentiate the first and last member of equation (3.1) and find for $j = 1$ that

$$(3.26) \quad \frac{\partial V_1^{(m)}}{\partial z} = r^m \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m+1} \int_0^1 (z+iat) \frac{\exp\{ikR^+(r,z;t)\}}{R^+(r,z;t)} f_1^{(m)}(t) dt$$

$$-r^m \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m+1} \int_0^1 (z-iat) \frac{\exp\{ikR^+(r,z;t)\}}{R^+(r,z;t)} f_1^{(m)}(t) dt;$$

owing to the identity

$$(3.27) \quad \frac{\partial}{\partial z} \left(\frac{\exp\{ikR^\pm\}}{R^\pm} \right) = \frac{1}{z \pm iat} \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\exp\{ikR^\pm\}}{R^\pm} \right).$$

Letting z approach zero through positive values, and making use of the limiting behavior of $R^\pm(r,z;t)$ [see equations (3.5) and (3.5')] we find that in $\mathcal{A}^{16}$

$$\begin{aligned}
 (3.28) \quad r^{-m} \frac{\partial V_1^{(m)}}{\partial z} &= 2i \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m+1} \int_0^{r/a} \frac{\cos(\alpha \sqrt{(r/a)^2 - t^2})}{\sqrt{(r/a)^2 - t^2}} t f_1^{(m)}(t) dt \\
 &\quad - 2 \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m+1} \int_0^{r/a} \frac{\sin(\alpha \sqrt{(r/a)^2 - t^2})}{\sqrt{(r/a)^2 - t^2}} t f_1^{(m)}(t) dt \\
 &\quad - 2 \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m+1} \int_{r/a}^1 \frac{\sinh(\alpha \sqrt{t^2 - (r/a)^2})}{\sqrt{t^2 - (r/a)^2}} t f_1^{(m)}(t) dt
 \end{aligned}$$

Since $\sin(\alpha \sqrt{(r/a)^2 - t^2}) / \sqrt{(r/a)^2 - t^2}$, $0 \leq t^2 \leq (r/a)^2$, is the continuation of $\sinh(\alpha \sqrt{t^2 - (r/a)^2}) / \sqrt{t^2 - (r/a)^2}$ when $t^2 \geq (r/a)^2$, we conclude that for $0 \leq r \leq a$

$$\begin{aligned}
 (3.29) \quad &\left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m+1} \int_0^{r/a} \frac{\cos(\alpha \sqrt{(r/a)^2 - t^2})}{\sqrt{(r/a)^2 - t^2}} t f_1^{(m)}(t) dt \\
 &- \frac{1}{i} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m+1} \int_0^1 \frac{\sinh(\alpha \sqrt{t^2 - (r/a)^2})}{\sqrt{t^2 - (r/a)^2}} t f_1^{(m)}(t) dt \\
 &= \frac{r^{-m}}{2i} \frac{dV_1^{(m)}(r, 0)}{dz}
 \end{aligned}$$

Now from equations (2.14) and (2.19) it follows that $\partial V_1^{(m)}(r, 0) / \partial z$, $0 \leq r \leq a$, has the form¹⁷

$$(3.30) \quad \frac{\partial V_1^{(m)}(r,0)}{\partial z} = k(kr)^m \sum_{\mu=0}^{\infty} h_{\mu}^{(m)}(kr)^{2\mu}$$

where $h_{\mu}^{(m)}$ is independent of k . Introducing

$$(3.31) \quad \eta = (r/a)^2,$$

$$(3.32) \quad \xi = t^2$$

into equation (3.29) and then setting $\xi = \eta \xi'$ in the first integrand, we find after dropping the prime on ξ' that

$$(3.33) \quad \left(\frac{d}{d\eta}\right)^{m+1} \int_0^1 \left[\frac{\sqrt{\eta} f_1^{(m)}(\sqrt{\eta\xi}) \cos[\alpha\sqrt{\eta(1-\xi)}]}{\sqrt{1-\xi}} - \frac{f_1^{(m)}(\sqrt{\xi}) \sinh[\alpha\sqrt{\eta-\xi}]}{i\sqrt{\eta-\xi}} \right] d\xi$$

$$= \frac{a^{m+1} \alpha^{m+1}}{i 2^{m+1}} \sum_{\mu=0}^{\infty} h_{\mu}^{(m)} (\alpha^2 \eta)^{\mu}$$

Proceeding in an analogous fashion with $V_2^{(m)}(r,z)$ we note from equations (2.14) and (2.19) that

$$(3.34) \quad V_2^{(m-1)} \Big|_{z \downarrow 0} = k^{m-1} r^{|m-1|} \sum_{\mu=0}^{\infty} p_{\mu}^{(m)} (kr)^{2\mu}, \quad 0 \leq r \leq a$$

where $p_{\mu}^{(m)}$ is independent of k and find that $f_2^{(m-1)}$ satisfies the following integral relation

$$(3.35) \quad \frac{d^{|m-1|}}{d\eta} \int_0^1 \left[\frac{f_2^{(m-1)}(\sqrt{\eta\xi}) \cos[\alpha\sqrt{\eta(1-\xi)}]}{\sqrt{\xi} \sqrt{1-\xi}} - \frac{f_2^{(m-1)}(\sqrt{\xi}) \sinh(\alpha\sqrt{\eta-\xi})}{i\sqrt{\xi} \sqrt{\eta-\xi}} \right] d\xi$$

$$= \frac{a^m \alpha^{m-1}}{2^{|m-1|}} \sum_{\mu=0}^{\infty} p_{\mu}^{(m)} (\eta \alpha^2)^{\mu}$$

for¹⁸ $m = 1, 2, \dots$

Equations (3.33) and (3.35) are the integral relations mentioned in the introductory section. Note that they follow from the fundamental representations (3.1) in essentially one elementary step.

Since equations (3.33) and (3.35) are basically of the same form it will suffice to explain how equation (3.33) is solved for $f_1^{(m)}(t)$.

We begin by assuming that α is small and seek a solution $f_1^{(m)}(t)$ of equation (3.33) having the form

$$\begin{aligned}
 (3.36) \quad f_1^{(m)}(t) &= t(1-t^2)^m \alpha^{m+1} \sum_{\mu=0}^{\infty} B_{\mu}^{(m)}(t) \alpha^{\mu} \\
 &= \eta^{1/2} (1-\eta) \alpha^{m+1} \sum_{\mu=0}^{\infty} B_{\mu}^{(m)}(\sqrt{\eta}) \alpha^{\mu} = f_1^{(m)}(\sqrt{\eta}) \\
 &\quad (\eta = t^2 \text{ here})
 \end{aligned}$$

where $B_{\mu}^{(m)}(t)$ is a polynomial which is even in t of degree at most¹⁹ μ .

It has been demonstrated by J. Bazer and H. Hochstadt [7, pp. 17-19] that a solution of this form exists and is unique for all sufficiently small²⁰ α , the power series converging for all t such that $|t| < 1 + \Delta/a$ [see the remarks following equation (2.20) and preceding equation (3.3)]. This being the case, we may substitute the expression for $f_1^{(m)}(\sqrt{\eta})$ into equation (3.33), expand in powers of α and equate coefficients of like powers. We then find that we are able to calculate the $B_{\mu}^{(m)}(t)$ recursively. To convey the basic idea of the procedure let us find $B_0^{(m)}$. Since the degree of $B_0^{(m)}$ is at most zero, $B_0^{(m)}$ must be a constant. Equating coefficients of α^{m+1} we obtain from equation (3.33)

$$(3.37) \quad B_0^{(m)} \frac{d^{m+1}}{d\eta^{m+1}} \int_0^1 \frac{\eta(1-\eta\xi)^m}{\sqrt{1-\xi}} d\xi = \frac{a^{m+1}}{i 2^{m+1}} h_0^{(m)}$$

owing to the fact that the term involving $\sinh[\alpha\sqrt{\eta-\xi}]$ is of order α^{m+2} .

From this relation it follows directly that

$$(3.38) \quad B_0^{(m)} = \frac{(-1)^m a^{m+1} h_0^{(m)}}{2^{m+1} i \sqrt{\pi} \Gamma(m+\frac{3}{2})}$$

The corresponding expressions for $B_2^{(m)}(t)$, $B_3^{(m)}(t)$, $B_4^{(m)}(t)$ are²¹

$$B_1^{(m)}(t) \equiv 0,$$

$$B_2^{(m)}(t) = \frac{(-1)^m a^{m+1}}{i \sqrt{\pi} 2^{m+1} \Gamma(m+\frac{5}{2})} \left\{ m \left[h_1^{(m)} - \frac{h_0^{(m)}}{4(m+\frac{1}{2})} \right] + t^2 \left[h_1^{(m)} + \frac{h_0^{(m)}}{4} \right] \right\}$$

$$B_3^{(m)}(t) = - \frac{2ah_0(0)}{9\pi^2} \delta_{m0},$$

$$(3.39) \quad B_4^{(m)}(t) = \frac{(-1)^m a^{m+1}}{i \sqrt{\pi} 2^{m+1} \Gamma(m+\frac{7}{2})} \left\{ m \left[(m+1)h_2^{(m)} - \frac{(m+1)h_1^{(m)}}{4(m+\frac{1}{2})} + \frac{2m^2-m+3}{64(m-\frac{1}{2})(m+\frac{1}{2})^2} h_0^{(m)} \right] \right. \\ \left. + mt^2 \left[2h_2^{(m)} + \frac{1}{4}h_1^{(m)} - \frac{h_0^{(m)}}{16(m+\frac{1}{2})} \right] \right. \\ \left. + t^4 \left[2h_2^{(m)} + \frac{1}{4}h_1^{(m)} + \frac{1}{32}h_0^{(m)} \right] \right\}$$

Similarly, we assume that $r_2^{(m-1)}(t)$ has the form

$$(3.40) \quad r_2^{(m-1)}(t) = (1-t^2)^{|m-1|} \alpha^{m-1} \sum_{\mu=0}^{\infty} c_{\mu}^{(m-1)}(t) \alpha^{\mu}$$

where $c_{\mu}^{(m-1)}(t)$ is an even polynomial of degree at most μ and find that equation (3.35) implies for $m \geq 1$

$$c_0^{(m-1)}(t) = \frac{(-1)^{m-1} a^m p_0(m)}{\sqrt{\pi} 2^{m-1} \Gamma(m - \frac{1}{2})} ,$$

$$c_1^{(m-1)}(t) = \frac{2ap_0(1)}{\pi 2_i} \delta_{1m} ,$$

$$c_2^{(m-1)}(t) = \frac{(-1)^{m-1} a^m}{\sqrt{\pi} 2^{m-1} \Gamma(m + \frac{1}{2})} \left\{ (m-1) \left[p_1(m) - \frac{p_0(m)}{4(m-\frac{3}{2})} \right] + t^2 \left[p_1(m) + \frac{1}{4} p_0(m) \right] \right\}$$

$$(3.41) \quad - \frac{4a p_0(1)}{\pi 3} \delta_{1m} ,$$

$$c_3^{(m-1)}(t) = \frac{2a}{\pi 2_i} \left[p_0(1) \left(\frac{1}{9} - \frac{4}{\pi 2} \right) + \frac{2}{3} p_1(1) + \frac{5t^2}{6} p_0(1) \right] \delta_{1m}$$

$$+ \frac{4a^2 p_0(2)}{9\pi 2_i} \delta_{2m} ,$$

and for $m = 0$ that

$$(3.42) \quad c_{\mu}^{(-1)}(t) = a^2 c_{\mu}^{(1)}(t) \quad \mu = 0, 1, 2, \dots$$

(It is to be understood here that the $p_{\mu}(1)$ which appear in the expressions for the $c_{\mu}^{(1)}(t)$ in equation (3.42) are to be replaced by $p_{\mu}(0)$.)

3.3 Ensuring the Proper Edge Behavior

As a mere glance at the form of $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ reveals [cf. equations (3.36) and (3.40)], the edge conditions (3.3) are satisfied. We have yet, however, to satisfy condition (3.4). To this end, [cf. references [8], [9], [10]] we recall that in the aperture $\mathcal{A}$ [see equation (2.14)]

$$(3.43) \quad \frac{dV_1^{(m)}}{dz} = c_m k(kr)^m - \frac{i}{k} \frac{d^2 V_0^{(m)}}{dz^2} ,$$

$$V_2^{(m-1)} = -mc_m (kr)^{m-1} - ik^{-1} r^{-m} \frac{\partial}{\partial r} \left(r^m V_0^{(m)} \right) ,$$

and note that each term in the right-hand members of these equations is of the form assumed in equations (3.30) and (3.34), respectively, and that c_m is an arbitrary constant. Next, we write

$$(3.44) \quad f_1^{(m)}(t) = c_m f_{1,1}^{(m)}(t) + f_{1,2}^{(m)}(t)$$

$$(3.45) \quad f_2^{(m-1)}(t) = c_m f_{2,1}^{(m-1)}(t) + f_{2,2}^{(m-1)}(t) ;$$

where we suppose that $f_{1,1}^{(m)}(t)$ and $f_{1,2}^{(m)}(t)$ are solutions of (3.29) with $k^{m+1}/2i$ and $(-\partial^2 V_0^{(m)}/\partial z^2)/(2k r^m)$ as inhomogeneous terms; we define

$f_{2,1}^{(m-1)}(t)$ and $f_{2,2}^{(m-1)}(t)$ with respect to the analog of equation (3.29)

in a similar manner. In this way, we obtain four functions $f_{1,j}^{(m)}$, $f_{2,j}^{(m-1)}$,

$j = 1,2$ with the following properties: i) $f_{1,1}^{(m)}(t)$, $f_{1,2}^{(m)}(t)$ have the

same form as the right member of equation (3.36) while $f_{2,1}^{(m-1)}(t)$ and $f_{2,2}^{(m-1)}(t)$

have the same form as the right member of (3.40); ii) $f_{1,j}^m(t)$ and $f_{2,j}^{(m-1)}(t)$, $j = 1, 2$, are independent of c_m .

To satisfy the edge condition (3.4) we now substitute the above expressions for $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$ into equation (3.4) and solve for c_m . The result is ²²

$$(3.46) \quad c_m = \frac{-ia\alpha D^{m-1} f_{2,2}^{(m-1)}(1) + \alpha D^m f_{1,2}^{(m)}(1)}{ia\alpha D^{m-1} f_{2,1}^{(m-1)}(1) - \alpha D^m f_{1,1}^{(m)}(1)}, \quad m = 0, 1, 2, \dots$$

From the definitions of $f_{2,1}^{(m-1)}(t)$ and $f_{1,1}^{(m)}(t)$ it follows that $\alpha D^{m-1} f_{2,1}^{(m-1)}(1) = \mathcal{O}[\alpha^m]$ and that $\alpha D^m f_{1,1}^{(m)}(1) = \mathcal{O}[\alpha^{m+2}]$ and consequently that the denominator of equation (3.46) does not vanish when α is sufficiently small. This fact also justifies expanding the reciprocal of the denominator by the binomial theorem in powers of $D^m f_{1,1}^{(m)}(1) / (ia\alpha D^{m-1} f_{2,1}^{(m-1)}(1))$. In this way, we find that for $m \geq 1$

$$(3.47) \quad c_m = \frac{2}{i} h_0^{(0)}(m) + \frac{\alpha^2}{2i(m - \frac{1}{2})} \left[4(m+1)h_1^{(0)}(m) + \frac{h_0^{(2)}(m)}{(m + \frac{1}{2})} + \frac{4h_0^{(0)}(m)}{2m+1} \right] \\ + \frac{4\alpha^3 \delta_{1m}}{3\pi} [8h_1^{(0)}(1) + 2h_0^{(0)}(1) + h_0^{(2)}(1)] \\ + \mathcal{O}[\alpha^4]$$

while for $m = 0$ we find that

$$(3.48) \quad c_0 = 4ia^2 h_1^{(0)}(0) + \mathcal{O}[\alpha^2]$$

where the quantities $h_0^{(0)}(m)$, $h_0^{(2)}(m)$ are defined in equation (2.19).

Similarly, we find in the case of H-mode excitation that for $m \geq 1$

$$(3.49) \quad \bar{c}_m = 2\alpha^2 h_o^{(1)}(m) \left[\frac{1}{l_{lm}^2 - 1} - \frac{2\alpha \delta_{lm}}{3\pi i} \right] + \mathcal{O}(\alpha^4)$$

while for $m = 0$

$$(3.50) \quad \bar{c}_o = h_o^{(1)}(0) + \frac{2}{3} \alpha^2 h_1^{(1)}(0) + \mathcal{O}[\alpha^3].$$

4. Explicit Results for Small Values of $\alpha = ka$

4.1 General Excitation. In what follows the symbols $h_\mu^{(0)}$, $h_\mu^{(1)}$, and

$h_\mu^{(2)}$ denote the constants appearing in equation (2.19). We distinguish between E-mode and corresponding H-mode quantities by placing an overbar on the latter. We define the symbol ϵ by

$$(4.1) \quad \epsilon = \sqrt{a^2 - r^2}.$$

The transmission coefficient t_m corresponding to an incident field $(\underline{E}_0^{(m)}, \underline{H}_0^{(m)})$ [see equations (2.4) and (2.5)] may be defined as the average power radiated to infinity in a period on the shadow side of the screen divided by the average power in the incident wave which enters the shadow side through the aperture in a period. It is easily shown, with the aid of equations (2.16) and (2.17), that for $m \geq 1$

$$(4.2) \quad t_m = \frac{\pi \alpha^4 \int_0^\pi \sin \theta \left\{ |A_2^{(m-1)}(\theta)|^2 + |\cos \theta A_2^{(m-1)}(\theta) - \sin \theta A_1^{(m)}(\theta)|^2 \right\} d\theta}{a^4 \text{Re} \int_0^{2\pi} \int_0^a \underline{E}_0^{(m)} \wedge \underline{H}_0^{(m)*} \cdot \underline{e}_z r dr d\phi}$$

$A_1^{(m)}(\theta)$ and $A_2^{(m-1)}(\theta)$ being defined as in equations (3.14) and (3.15). In equation (4.2) the asterisk denotes complex conjugation and $\underline{e}_z$ denotes the unit vector in the z-direction. This formula applies to H-mode as well as to E-mode excitation. When $m = 0$ we find for E-mode excitation that

$$(4.3) \quad t_0 = \frac{2\pi \alpha^4 \int_0^\pi \sin \theta \left\{ \cos^2 \phi_0^{(0)} |A_2^{(-1)}(\theta) \cos \theta - A_1^{(0)}(\theta) \sin \theta|^2 + \sin^2 \phi_0^{(0)} |A_2^{(-1)}(\theta)|^2 \right\} d\theta}{a^4 \text{Re} \int_0^{2\pi} \int_0^a \underline{E}_0^{(0)} \wedge \underline{H}_0^{(0)*} \cdot \underline{e}_z r dr d\phi}$$

The same formula holds for H-mode excitation when $\phi_0^{(0)}$ is replaced by $\bar{\phi}_0^{(0)}$.

In this subsection, we list the first terms in the power series expressions for various quantities of interest - general E- and H-mode excitation is assumed. (see the neighborhood of equation (2.19)). In the next subsection we specialize these general formulas to the case of plane wave excitation in which case the constants $h_{\mu}^{(j)}$, and $\bar{h}_{\mu}^{(j)}$, $\mu = 0,1,2,\dots$ $j = 0,1,2$ are given by the expressions listed in equation (2.24). The unit vectors $\underline{e}_r, \underline{e}_{\phi}, \underline{e}_{\theta}, \underline{e}_R$ appearing in the expressions below are defined at the beginning of Section 2; $\underline{e}_x, \underline{e}_y$ and $\underline{e}_z$ denote the unit vectors parallel to the x, y and z axes respectively.

(E-mode) $m \geq 1$

In the aperture $\mathcal{A}$, we find that

$$(4.4) \quad V_1^{(m)} = \frac{\sqrt{\pi} i [(2m)!]^2 (2m+1) [h_o^{(2)}(m) + 2h_o^{(0)}(m)] r^m \epsilon \alpha^{m+1}}{2^{4m+1} a^{m+1} m! \Gamma(m + \frac{1}{2}) \Gamma^2(m + \frac{3}{2})} + \mathcal{O}[\alpha^{m+3}],$$

$$(4.5) \quad V_2^{(m-1)} = r^{m-1} \alpha^{m+1} \left[\frac{2i(m+1)h_1^{(0)}(m) \{m \epsilon^2 + \frac{1}{2} r^2\}}{a^{m+1} (m - \frac{1}{2})} + \frac{mi \{h_o^{(2)}(m) + 2h_o^{(0)}(m)\}}{2a^{m-1} (m^2 - \frac{1}{4})} \right] + \delta_{1m} \mathcal{O}[\alpha^3],$$

where

$$(4.6) \quad \delta_{1m} = \begin{cases} 1 & \text{when } m = 0 \\ 0 & \text{when } m \neq 0 \end{cases}$$

The tangential component of the electric field in the aperture is

$$(4.7) \quad \tilde{E}_t = \frac{(-1)^m m(m+1)(2m-1) [(2m-2)!]^2 r^{m-1}}{2^{4m-3} a^{m+2} (m-1)! \sqrt{\pi} \Gamma(m + \frac{3}{2})} \epsilon \alpha^{m+2} \times$$

$$\left\{ h_o^{(2)}(m) + (8m+4)h_1^{(o)}(m) + 2h_o^{(o)}(m) \right\} \{ \sin m\varphi_{m\tilde{r}} e_{m\tilde{r}} + \cos m\varphi_{m\tilde{p}} e_{m\tilde{p}} \} \\ + \left\{ \frac{h_o^{(2)}(m) + 2h_o^{(o)}(m)}{m+1} \right\} \sin m\varphi_m \frac{r^2}{\epsilon^2} \tilde{e}_r \left] + \mathcal{O}[\alpha^{m+4}]. \right.$$

The far field behavior of $\tilde{E}$ is [we remind the reader that $H = \tilde{e}_R \wedge \tilde{E}$ where $R \rightarrow \infty$. See equation (2.16)]:

$$(4.8) \quad \tilde{E} \sim \frac{(-1)^m \sqrt{\pi} i^m (\sin \theta)^{m-1} \alpha^{2m+2}}{2^{3m-2} a \Gamma(m + \frac{1}{2}) \Gamma(m + \frac{3}{2})} \left[\frac{(m+1)(2m)! h_1^{(o)}(m)}{\Gamma(m + \frac{1}{2})} \right. \\ \left. + \frac{m(2m-2)! \{ h_o^{(2)}(m) + 2h_o^{(o)}(m) \}}{\Gamma(m + \frac{1}{2})} \right] \left[\sin m\varphi_{m\tilde{\theta}} e_{m\tilde{\theta}} + \cos \theta \cos m\varphi_{m\tilde{\varphi}} e_{m\tilde{\varphi}} \right] \frac{e^{ikR}}{R} \\ + \delta_{lm} \mathcal{O}[\alpha^5] \quad (R \rightarrow \infty).$$

Finally, the transmission coefficient is

$$(4.9) \quad t_m = \frac{-\pi^2 (m+1)(2m+1)(2m-1)! [(2m-2)!]^2}{im2^{8m-4} |h_o^{(o)}(m)|^2 \Gamma^4(m + \frac{1}{2}) \Gamma^4(m + \frac{3}{2})} \left(\frac{\alpha^{2m+2}}{\beta^{(m)}} \right) \times \\ [2m(2m-1)(m+1)h_1^{(o)}(m) + m(m - \frac{1}{2}) \{ h_o^{(2)} + 2h_o^{(o)}(m) \}]^2 + \mathcal{O}[\alpha^{2m+4}]$$

When $m = 0$, the corresponding results are:

$$(4.10) \quad V_1^{(0)} = \frac{2\alpha\epsilon}{\pi i a} [4a^2 h_1^{(0)}(0) - h_o^{(2)}(0)] + O[\alpha^3],$$

$$(4.11) \quad V_2^{(-1)} = - \frac{2i\alpha r h_1^{(0)}(0)}{a} + O[\alpha^3],$$

$$(4.12) \quad \begin{aligned} E_t = & \frac{+8h_1^{(0)}(0) \sin \phi_o^{(0)} r a^2 \underline{e}_r}{\pi a^2 \epsilon} \\ & + \frac{[8(a^2-1)h_1^{(0)}(0) - 2h_o^{(2)}(0)]}{\pi a^2 \epsilon} \cos \phi_o^{(0)} r a^2 \underline{e}_\phi \\ & + O[\alpha^4], \end{aligned}$$

$$(4.13) \quad \begin{aligned} E \sim & \frac{2\alpha^4 \sin \theta}{3\pi a} [\cos \theta \{4(1-a^2)h_1^{(0)}(0) + h_o^{(2)}(0)\} \cos \phi_o^{(0)} \underline{e}_\phi \\ & - 4h_1^{(0)}(0) \sin \phi_o^{(0)} \underline{e}_\theta] \frac{e^{ikR}}{R} + O[\alpha^6] \quad (R \rightarrow \infty), \end{aligned}$$

$$(4.14) \quad \begin{aligned} t_o = & - \frac{16a}{45\pi^2 i} \frac{[|4(1-a^2)h_1^{(0)}(0) + h_o^{(2)}(0)|^2 \cos^2 \phi_o^{(0)} + 80|h_1^{(0)}(0)|^2 \sin^2 \phi_o^{(0)}]}{|h_1^{(0)}(0)|^2 \sin^2 \phi_o^{(0)}} \left(\frac{\alpha^2}{\beta} \right) \\ & + O[\alpha^4] \end{aligned}$$

(H-Mode) $m \geq 1$

In the aperture $\mathcal{A}$ we find that

$$(4.15) \quad \bar{V}_1^{(m)} = \frac{\sqrt{\pi} [(2m)!]^2 (2m+1) \bar{h}_o^{(1)}(m) r^m \epsilon^{\alpha^{m+1}}}{2^{4m+1} m! a^{m+1} \Gamma(m + \frac{1}{2}) \Gamma^2(m + \frac{3}{2})} + \mathcal{O}[\alpha^{m+3}],$$

$$(4.16) \quad \bar{V}_2^{(m-1)} = \frac{m \bar{h}_o^{(1)}(m) r^{m-1} \alpha^{m+1}}{2 a^{m-1} (m^2 - \frac{1}{4})} + \delta_{lm} \mathcal{O}[\alpha^3],$$

The tangential component of the electric field in the aperture is

$$(4.17) \quad \bar{E}_t = \frac{-im [(2m-2)!]^2 (2m-1)(m+1) \sqrt{\pi} \bar{h}_o^{(1)}(m) r^{m-1} \epsilon^{\alpha^{m+2}}}{2^{4m-3} a^{m+2} (m-1)! \Gamma(m - \frac{1}{2}) \Gamma(m + \frac{1}{2}) \Gamma(m + \frac{3}{2})} \times \\ \left[\cos m\bar{\phi}_m \bar{e}_\varphi + \sin m\bar{\phi}_m \left(1 + \frac{r^2}{(m+1)\epsilon^2} \right) \bar{e}_r \right] + \mathcal{O}[\alpha^{m+4}],$$

The far field behavior of $\bar{E}$ is

$$(4.18) \quad \bar{E} \sim \frac{(-1)^m \sqrt{\pi} i^{m-1} m (2m-2)! \bar{h}_o^{(1)}(m) (\sin \theta)^{m-1} \alpha^{2m+2}}{2^{3m-2} a \Gamma(m - \frac{1}{2}) \Gamma(m + \frac{1}{2}) \Gamma(m + \frac{3}{2})} \times \\ \left[\sin m\bar{\phi}_m \bar{e}_\theta + \cos \theta \cos m\bar{\phi}_m \bar{e}_\varphi \right] \frac{e^{ikR}}{R} \quad (R \rightarrow \infty).$$

The transmission coefficient is

$$(4.19) \quad \bar{t}_m = \frac{\pi^2 m(m+1)(2m+1)(2m-1)! [(2m)!]^2 |\bar{h}_o^{(1)}(m)|^2}{i 2^{8m} |\bar{h}_o^{(0)}(m)|^2 \Gamma^2(m + \frac{1}{2}) \Gamma^6(m + \frac{3}{2})} \cdot \frac{\alpha^{2m+2}}{(\bar{\beta}^{(m)})^*} \\ + \mathcal{O}[\alpha^{2m+4}]$$

When $m = 0$ the corresponding results are:

$$(4.20) \quad \bar{V}_1^{(0)} = \frac{-8\bar{h}_1^{(1)}(0) \epsilon^3 \alpha^3}{9\pi a^3} + \mathcal{O}[\alpha^4],$$

$$(4.21) \quad \bar{V}_2^{(-1)} = 0,$$

$$(4.22) \quad \bar{E}_t \approx \frac{-8i\bar{h}_1^{(1)}(0) \cos^2 \varphi_o^{(0)} r \epsilon \alpha^4 e_{\varphi}}{3\pi a^4} + \mathcal{O}[\alpha^5],$$

$$(4.23) \quad \bar{E} \sim \frac{-8\bar{h}_1^{(1)}(0) \cos^2 \varphi_o^{(0)} \cos \theta \sin \theta \alpha^6}{45\pi i a} \frac{e^{ikR}}{R} + \mathcal{O}[\alpha^7] \quad (R \rightarrow \infty)$$

$$(4.24) \quad \bar{t}_o = \frac{256a^4 |\bar{h}_1^{(1)}(0)|^2 \alpha^6}{10,125\pi^2 i |\bar{h}_1^{(0)}(0)|^2 (\bar{\beta}^{(0)})^*} + \mathcal{O}(\alpha^7).$$

(4.2) Plane Wave Excitation. We wish to check our results for $m = 0$ and $m = 1$ against those collected by C. J. Bouwkamp in references [3a] and [3b]. To this end, we note that Bouwkamps' angle θ_o corresponds to our angle γ [see the neighborhood of equation (2.21)] and that Bouwkamps' plane wave is the linear combination

$$-k^{-2} \cos \phi_o(\underline{E}_o, \underline{H}_o) - k^{-2} \sin \phi_o(\bar{\underline{E}}_o, \bar{\underline{H}}_o)$$

of the plane wave fields $(\underline{E}_o, \underline{H}_o)$ and $(\bar{\underline{E}}_o, \bar{\underline{H}}_o)$ defined in equations (2.21). Setting $m = 0$ and $m = 1$ in the above E-mode and H-mode formulas, inserting for the constants $h_\mu^{(j)}(m)$ and $\bar{h}_\mu^{(j)}(m)$ ($m = 1, 2$) appearing there the

expressions in the right members of equations (2.25) and finally forming the appropriate linear combination of the four fields thus obtained, we arrive at the results listed below.

The tangential component of the electric field in the aperture is

$$\begin{aligned}
 (4.25) \quad \tilde{E}_t = & \frac{-2 \cos \phi_0 \sin \theta_0 r \tilde{e}_r}{\pi \sqrt{a^2 - r^2}} - \frac{8i\alpha}{a} \frac{1}{\sqrt{a^2 - r^2}} \left[\sin \phi_0 \cos \theta_0 (a^2 - \frac{1}{2}r^2) \sin \phi \right. \\
 & \left. + \cos \phi_0 \left\{ (a^2 - \frac{1}{2}r^2) + (r^2 - \frac{1}{2}a^2) \sin^2 \theta_0 \cos \phi \right\} \right] \tilde{e}_r \\
 & + \frac{8i\alpha \sqrt{a^2 - r^2}}{a} \left[-\sin \phi_0 \cos \theta_0 \cos \phi + \cos \phi_0 (1 - \frac{1}{2} \sin^2 \theta_0) \sin \phi \right] \tilde{e}_\phi \\
 & + O[\alpha^2].
 \end{aligned}$$

The far-field behavior of the electric field is

$$\begin{aligned}
 \tilde{E} \sim \frac{k^2 a^3}{3\pi} \tilde{e}_R \wedge [& (-4 \sin \phi_0 \cos \theta_0 + 2 \cos \phi_0 \sin \theta_0 \sin \theta \sin \phi) \tilde{e}_x \\
 & + (4 \cos \phi_0 - 2 \cos \phi_0 \sin \theta_0 \sin \theta \cos \phi) \tilde{e}_y] \frac{e^{ikR}}{R}
 \end{aligned}$$

The formula for transmission cross-section is

$$\sigma = \frac{64a^2 \alpha^4}{27\pi} \left[\sin^2 \phi_0 \cos^2 \theta_0 + \cos^2 \phi_0 + \frac{1}{4} \cos^2 \phi_0 \sin^2 \theta_0 \right] + O[\alpha^6],$$

where the transmission cross-section is the numerator of the transmission coefficient.

The results above are precisely those given by Bouwkamp in [3a,b].

Appendix I

In the aperture the r and ϕ components of the scattered magnetic field $\underline{H}$ are

$$(I.1) \quad H_r = \cos m\phi_m \left\{ k^2 V_2^{(m-1)} + \frac{\partial \Gamma^{(m-1)}}{\partial r} + \frac{\partial}{\partial r} \left(\frac{\partial V_1^{(m)}}{\partial z} \right) \right\},$$

$$H_\phi = -\sin m\phi_m \left\{ k^2 V_2^{(m-1)} + \frac{m}{r} \Gamma^{(m-1)} + \frac{m}{r} \frac{\partial V_1^{(m)}}{\partial z} \right\},$$

where

$$(I.2) \quad \Gamma^{(m-1)} = \frac{1}{r} \frac{\partial}{\partial r} \left(r V_2^{(m-1)} \right) - \frac{m}{r} V_2^{(m-1)}.$$

These formulas follow directly from equations (2.6) and (2.9). We consider E-mode excitation and H-mode excitation separately

E-mode Excitation: In this case the condition that $\underline{H}_t = \underline{H}_{o,t}$ in $\mathcal{Q}$ and, equations (I.1), (I.2) and (2.4b) imply

$$(I.3) \quad -\frac{ikm}{r} V_o^{(m)} = k^2 V_2^{(m-1)} + \frac{\partial \Gamma^{(m-1)}}{\partial r} + \frac{\partial}{\partial r} \left(\frac{\partial V_1^{(m)}}{\partial z} \right),$$

$$(I.4) \quad -ik \frac{\partial V_o}{\partial r} = k^2 V_2^{(m-1)} + \frac{m}{r} \Gamma^{(m-1)} + \frac{m}{r} \frac{\partial V_1^{(m)}}{\partial z}.$$

Subtracting equation (I.4) from (I.3) we get

$$(I.5) \quad \left(\frac{\partial}{\partial r} - \frac{m}{r} \right) \left[\frac{\partial V_1^{(m)}}{\partial z} + \Gamma^{(m-1)} - ik V_o^{(m)} \right] = 0,$$

from which we infer immediately that

$$(I.8) \quad \frac{\partial V_1^{(m)}}{\partial z} + \Gamma^{(m-1)} - ikV_0^{(m)} = c_m k(kr)^m.$$

where c_m is an arbitrary constant. Substituting the expression of $\partial V_1^{(m)}/\partial z + \Gamma^{(m-1)}$ obtained from this equation into equation (I.4) and combining terms in an obvious manner we find that

$$(I.7) \quad V_2^{(m-1)} = \frac{-i}{k} r^{-m} \frac{\partial}{\partial r} \left(r^m V_0^{(m)} \right) - mc_m (kr)^{m-1}.$$

Substituting in turn this expression for $V_2^{(m-1)}$ into the right member of equation (I.2), putting the resulting expression for $\Gamma^{(m-1)}$ into equation (I.6) and noting that

$$(I.8) \quad \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_0^{(m)}}{\partial z} \right) + \left[k^2 - \left(\frac{m}{r} \right)^2 \right] V_0^{(m)} = - \frac{\partial^2 V_0^{(m)}}{\partial z^2},$$

[see equation (2.4d)] we get

$$(I.9) \quad \frac{\partial V_1^{(m)}}{\partial z} = c_m k(kr)^m - \frac{i}{k} \frac{\partial^2 V_0^{(m)}}{\partial z^2}.$$

Equations (I.7) and (I.9) will be recognized as the conditions listed in equation (2.14).

H-mode Excitation: Employing equations (I.1), (I.2) and (2.5b), we obtain the analogs of (I.3) and (I.4) [See footnote 7]

$$(I.10) \quad - \frac{\partial}{\partial r} \left(\frac{\partial \bar{V}_0^{(m)}}{\partial z} \right) = k^2 \bar{V}_2^{(m-1)} + \frac{\partial}{\partial r} \left(\frac{\partial \bar{V}_1^{(m)}}{\partial z} \right) + \frac{\partial \bar{\Gamma}^{(m-1)}}{\partial r}$$

$$(I.11) \quad \frac{\partial \bar{V}_0^{(m)}}{\partial z} = -k^2 \frac{r}{m} \bar{V}_2^{(m-1)} - \frac{\partial \bar{V}_1^{(m)}}{\partial z} - \frac{\partial \bar{\Gamma}^{(m-1)}}{\partial r}$$

where

$$\bar{\Gamma}^{(m-1)} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \bar{V}_2^{(m-1)} \right) - \frac{m}{r} \bar{V}_2^{(m-1)}$$

Differentiating equation (I.11) with respect to r and adding the resulting equation to (I.10) we get

$$(I.12) \quad \frac{\partial}{\partial r} \left(r \bar{V}_2^{(m-1)} \right) - m \bar{V}_2^{(m-1)} = 0$$

This equation implies

$$(I.13) \quad \bar{V}_2^{(m-1)} = m \bar{c}_m (kr)^{m-1}$$

where $\bar{c}_m$ is an arbitrary constant. Substituting this expression for $\bar{V}_2^{(m-1)}$ into equation (I.11) we find that

$$(I.14) \quad \bar{V}_1^{(m)} = - \frac{\partial \bar{V}_0^{(m)}}{\partial z} - \bar{c}_m k (kr)^m,$$

Equations (I.13) and (I.14) will be recognized as those listed in (2.15).

FOOTNOTES

1. Marcuvitz's solution is outlined in our appendix of Bouwkamp's report.
2. K. A. Lur'e's paper is an outgrowth of a work by N. N. Lebedev and I. P. Skal'skaya [10] who considered the special case of normal incidence of a plane wave on a disc. Both Lur'e and Williams are concerned with the diffraction by a disc. The solution of the disc and aperture problems are of course related through Babinet's principle.
3. We are grateful to W. D. Collins for bringing to our attention the existence of yet another solution of the vector aperture problem by J. P. Benkard University of Michigan thesis entitled "On an axisymmetric vector boundary value problem with special reference to the circular disc."
4. Representations of the kind appearing in equation (1.1) were originally introduced by J. Bazer and A. Brown [10] for the case $m = 0$, by W. D. Collins [11] for the cases $m = 0, 1$ and by J. Bazer and H. Hochstadt [6] for general values of $m \geq 0$.
5. These integro-differential equations lead to Fredholm integral equations of the second kind having Jones-type kernels [see reference [13a]]. Such integral equations are useful in establishing the existence of $f_1^{(m)}(t)$ and $f_2^{(m-1)}(t)$; however, as explained in reference [7], they do not provide an effective method for calculating $f_1^{(m)}$ and $f_2^{(m-1)}$ when m is large even when ka is small.
6. It would be tedious but not impractical to obtain a few additional terms.
7. Wherever E and H-modes are discussed together and there is a risk of confusing E-mode quantities with H-mode quantities, we employ an overbar to distinguish the latter from the former as in equations (2.4), (2.5) and (2.14)-(2.15).

8. Various formulations of the edge conditions for the vector problem will be found in the papers by C. J. Bouwkamp [3a,b] J. Meixner [1b] and D. S. Jones [13b]. We employ here a formula due to N. Marcuvitz [See appendix of reference [3b]].
9. This assumption is employed to merely simplify the general formulas for the transmission coefficients [see equations (4.9),(4.14) and (4.19),(4.24)]; it is not used in any other calculation.
10. This section is patterned after Sections 2 and 3 of J. Bazer and H. Hochstadt [7].
11. See property (i) below.
12. When $j = 2$ and $m = -1$, we require that (ii) be satisfied for $(ka)f_2^{(-1)}(t)$.
13. Property (2) $(f_1^{(m)}(t))$ is regular in the domain $|t| \leq 1 + \Delta/a$, $\Delta > 0$ is used in this connection.
14. The limit of the integral term as z approaches zero is zero owing to the oddness of $f_1^{(m)}(t)$.
15. Equations (3.20)-(3.24) hold for all $m \geq 0$ except that the factor $1/a^m$ in the right members of (3.22) and (3.24) must be replaced by $1/a^2$ when $m = 0$.
16. Hereafter, we shall use the abbreviation $\alpha = ka$.
17. We ignore for the moment the presence of the undetermined constant c_m .
18. When $m = 0$ the factor a^m in the right member must be replaced by a^4 .
19. This restriction on the degree of $B_\mu^{(m)}(t)$ may be verified by substituting $f_1^{(m)}(\sqrt{\eta})$ into equation (3.33), expanding in power of α and comparing coefficients of like powers.
20. The radius of convergence of this power series has not been determined. For the case of plane wave excitation and $m = 0$ it is known to be at least $1/2$ [see reference [15]].
21. δ_{ij} denotes the Kronecker delta; $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.
22. We have multiplied the numerator and denominator of this fraction by $\alpha = ka$ to avoid having to discuss the case $m = 0$ separately.

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